

# BETTI SPLITTING FROM A TOPOLOGICAL AND COMPUTATIONAL POINT OF VIEW

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**ABSTRACT.** A Betti splitting  $I = J + K$  of a monomial ideal  $I$  ensures to recover the graded Betti numbers of  $I$  starting from those of  $J, K$  and  $J \cap K$ . In this paper, we formalize this condition for simplicial complexes, using Alexander duality. We investigate the issue of admitting or not a Betti splitting with a topological and computational approach. Inspired by the relevant class of shellable simplicial complexes, we study decompositions induced by the removal of a single facet. Moreover, we apply these results to manifolds, proving that orientability is a sufficient condition to ensure the existence of a Betti splitting of this kind. We improve the computational aspects of this subject; in particular, we present several useful algorithms and we introduce the notion of splitting probability.

## 1. INTRODUCTION

A fundamental tool to describe the structure of a homogeneous ideal  $I$  in a polynomial ring is given by the minimal graded free resolution of  $I$  and, in particular, by its graded Betti numbers  $\beta_{i,j}(I)$ . Dealing with ideals of large size, the retrieval of these algebraic invariants can be hard from a computational point of view, also in the case of monomial ideals. A common strategy to obtain the information on  $I$  is to decompose it into smaller ideals, in order to recover the invariants of  $I$  using the invariants of its pieces. Following this idea, originally introduced in [6] and developed in [8], a Betti splitting of a monomial ideal  $I$  consists of a suitable decomposition  $I = J + K$  of  $I$  ensuring the complete retrieval of the graded Betti numbers of  $I$  from the ones of  $J, K$  and  $J \cap K$ . The decomposition  $I = J + K$  is called a *Betti splitting* of  $I$  if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K), \text{ for all } i, j \in \mathbb{N}.$$

In this paper, we formalize this condition for simplicial complexes. Combinatorial structures arise in several concrete applications (for instance computational topology, data analysis and shape recognition). Our aim is to give a strong computational support for researches focused on Betti splitting: we implemented several algorithms in Macaulay2 [9] and Python to check, for instance, if a given decomposition of a simplicial complex is a Betti splitting, see <https://github.com/fugacci/Check-Decompositions.git>. We think that this can be extremely useful for further developments on this subject.

In several application domains, it is interesting to describe the topology of a geometric realization of a simplicial complex  $\Delta$ , in particular its homology. The remarkable Hochster's formula relates the graded Betti numbers of the Alexander dual ideal  $I_{\Delta}^*$  of  $\Delta$  and the reduced homology of suitable subcomplexes of  $\Delta$ ; in this way Betti splittings are related to the classical Mayer-Vietoris approach to solve homological problems. Reading the Betti splitting condition for  $I_{\Delta}^*$  from a purely topological point of view, we investigate topological

properties and features of a geometric realization of  $\Delta$  inducing a suitable Betti splitting for  $I_\Delta^*$ .

In this framework, we introduce the notion of *homological splitting* for a simplicial complex  $\Delta$ , see Definition 3.5. It corresponds to a decomposition of  $I_\Delta^*$  for which the previous equation on graded Betti numbers holds for  $j = n$  and for every  $i \in \mathbb{N}$ . Algorithm 1 checks if a given decomposition of  $\Delta$  is a homological splitting. Using this notion, we are able to prove a complete characterization of Betti splitting for simplicial complexes, pointing out also the intrinsic recursive nature of this tool, see Theorem 3.10. Algorithm 2 checks if a given decomposition of  $\Delta$  is a Betti splitting, taking advantage of the characterization given in this result.

Inspired by the case of shellable simplicial complexes, we study homological splittings induced by the removal of a single monomial of  $I_\Delta^*$ , introducing the notion of *essential facet* of a simplicial complex, see Definition 4.3. If there is at least an essential facet in  $\Delta$ , then a suitable homological splitting is ensured (Theorem 4.5) and the existence of essential facets of dimension equal to  $\dim(\Delta)$  is related to the non-vanishing of top homology (Theorem 4.8). As consequence, a simplicial complex with top-homology always admits a suitable Betti splitting (Corollary 4.11).

In the relevant case of simplicial manifolds without boundary, the homological splitting condition for the removal of a single facet induces a Betti splitting (Theorem 5.7). This yields to the existence of Betti splittings for orientable simplicial manifolds (Corollary 5.8) and for all simplicial manifolds if  $\text{char}(\mathbb{F}) = 2$  (Corollary 5.9). Moreover we are able to characterize orientability of a manifold in terms of Betti splittings (Remark 5.11).

In Section 6, we consider pathological simplicial complexes that do not admit Betti splitting, depending on the field. We investigate obstructions that prevent such kind of decomposition for these complexes (Proposition 6.2), introducing Algorithm 3. In the Example 6.3, we present the first example in literature of an ideal with characteristic-dependent resolution with a Betti splitting over every field. All our result are independent on the chosen triangulation of the considered manifolds. To formalize this statement, we introduce the notion of *Betti splitting probability*, see Definition 6.4. Several results of the paper can be given in terms of this new notion and it can be the starting point for future investigations, using Algorithm 4.

## 2. PRELIMINARIES

Let  $\mathbb{F}$  be a field,  $R = \mathbb{F}[x_1, \dots, x_n]$  be the polynomial ring on  $n$  variables with coefficients in  $\mathbb{F}$  and  $\mathfrak{m} = (x_1, \dots, x_n)$  its maximal homogeneous ideal. Let  $M$  be a finitely generated graded  $R$ -module. The *minimal graded free resolution* of  $M$  as  $R$ -module is a free resolution of  $M$  of the form

$$0 \rightarrow F_p \xrightarrow{\phi_p} \dots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \rightarrow 0$$

where  $F_i = \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{i,j}(M)}$ ,  $\phi_i$  are homogeneous maps and  $\text{Im}(\phi_i) \subseteq \mathfrak{m}F_{i-1}$  for each integer  $i$ . The invariants  $\beta_{i,j}(M) = \dim_{\mathbb{F}}(F_i)_j = \dim_F \text{Tor}_i(M, \mathbb{F})_j$  are the *graded Betti numbers* of  $M$ . Denote by  $\beta_i(M) = \sum_{j \in \mathbb{N}} \beta_{i,j}(M)$  the  $i^{\text{th}}$  *total Betti number* of  $M$ .

If  $M$  is graded over  $\mathbb{Z}^n$ , we can consider its multigraded resolution and its *multigraded Betti numbers*  $\beta_{i,\underline{a}}(M)$ , where  $\underline{a} \in \mathbb{Z}^n$ . Typical examples of multigraded modules are given by monomial ideals. Denote by  $\text{supp}(\underline{a})$  the set  $\{i : a_i \neq 0\}$ .

Given a monomial ideal  $I \subseteq R$ , we denote by  $G(I)$  the minimal system of monomial generators of  $I$ .

**Definition 2.1.** Let  $I, J$  and  $K$  be monomial ideals such that  $I = J + K$  and  $G(I)$  is the disjoint union of  $G(J)$  and  $G(K)$ . We say that  $J + K$  is a *Betti splitting* of  $I$  over  $\mathbb{F}$  if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K), \text{ for all } i, j \in \mathbb{N}.$$

This condition is equivalent to the vanishing of some maps between Tor modules, see [8, Proposition 2.1]. In particular, the induced maps  $\text{Tor}_i(J \cap K; \mathbb{F})_j \rightarrow \text{Tor}_i(J; \mathbb{F})_j \oplus \text{Tor}_i(K; \mathbb{F})_j$  must be zero, for all  $i, j \in \mathbb{N}$ . Using this condition, it is straightforward to prove that the Betti splitting condition for degrees is equivalent to the Betti splitting condition for multidegrees.

**Example 2.2.** Let  $I = (x_4x_5, x_1x_5, x_1x_3, x_1x_2) \subseteq \mathbb{F}[x_1, \dots, x_5]$ . Consider the splitting  $I = J + K$ , with  $J = (x_4x_5, x_1x_3)$  and  $K = (x_1x_5, x_1x_2)$ . It is not difficult to see that  $\beta_{1,4}(J) \neq 0$ , since  $J$  is a complete intersection. But  $\beta_{1,4}(I) = 0$ , since  $I$  is an ideal with linear resolution. Then, the considered splitting is not a Betti splitting. Instead, for instance,  $I = (x_4x_5, x_1x_5) + (x_1x_3, x_1x_2)$  is a Betti splitting of  $I$ .

In some cases, a special kind of splitting is considered for a monomial ideal  $I$ . Assume  $I, J$  and  $K$  are monomial ideals as in Definition 2.1 and  $K = (m)$  is generated by a single monomial  $m$ . We call the decomposition  $I = J + K$  a *facet splitting* if it is a Betti splitting.

In this paper, we study splitting properties of ideals associated to simplicial complexes. An *abstract simplicial complex*  $\Delta$  on  $n$  vertices is a collection of subsets of  $[n] = \{1, \dots, n\}$ , called *faces*, such that if  $F \in \Delta$ ,  $G \subseteq F$ , then  $G \in \Delta$ . A simplicial complex  $\Delta$  is completely determined by the collection  $\mathcal{F}(\Delta)$  of its *facets*, the maximal faces with respect to inclusion. We denote by  $\langle F_1, \dots, F_r \rangle$  the simplicial complex determined by the facets  $F_1, \dots, F_r$ .

A *simplex of dimension  $k$*  or  *$k$ -simplex* is the convex hull of  $k + 1$  points in general position in  $\mathbb{R}^n$ . Every abstract simplicial complex can be seen as a topological space  $|\Delta|$ : in fact the facets of  $\Delta$  can be realized as simplices in a suitable Euclidean real space in such a way that two simplices must intersect in a common subface (possibly the empty face). The *dimension*  $\dim(\Delta)$  of  $\Delta$  is the largest dimension of its simplices.

Let  $k \in \mathbb{N}$ . Denote by  $\tilde{H}_k(\Delta; \mathbb{F})$  the  $k^{\text{th}}$  simplicial reduced homology group of  $\Delta$  with coefficients in  $\mathbb{F}$ . It can be proved that the simplicial homology of  $\Delta$  is isomorphic to the singular homology of a geometric realization  $|\Delta|$ , see [11]. We denote by  $\tilde{\beta}_k(\Delta; \mathbb{F})$  the dimension as  $\mathbb{F}$ -vector space of  $\tilde{H}_k(\Delta; \mathbb{F})$ . Recall that  $\tilde{\beta}_{-1}(\Delta; \mathbb{F}) \neq 0$  if and only if  $\Delta = \{\emptyset\}$ ; in this case  $\tilde{\beta}_{-1}(\Delta; \mathbb{F}) = 1$ .

There are several ways to associate a squarefree monomial ideal to a simplicial complex  $\Delta$ . In the literature, the most studied ideal is the so-called *Stanley-Reisner ideal*. In this paper, we are interested in the Alexander dual ideal of  $\Delta$ , that can be viewed as the Stanley-Reisner ideal of a suitable complex  $\Delta^*$  (for more details see for instance [12]).

**Definition 2.3.** Let  $\Delta$  be a simplicial complex on  $[n]$ . The *Alexander dual ideal* of  $\Delta$  is the squarefree monomial ideal defined by

$$I_{\Delta}^* = (x_{\overline{F}} : F \in \mathcal{F}(\Delta)) \subseteq R = \mathbb{F}[x_1, \dots, x_n],$$

where  $x_{\overline{F}} = \prod_{i \in [n] \setminus F} x_i$ .

**Remark 2.4.** Given a squarefree monomial ideal  $I \subseteq R$  and fixing the number of variables of  $R$ , there is a *unique* simplicial complex  $\Delta$  such that  $I = I_{\Delta}^*$ .

In [13], M. Hochster proved a remarkable formula that is the main bridge between Combinatorics and Commutative Algebra: it gives a relation between the homology of suitable subcomplexes of a simplicial complex and the graded Betti numbers of its associated ideals. The version of Hochster's formula that we recall here is due to Eagon and Reiner [5].

Given a simplicial complex  $\Delta$ , we define the *link* of a face  $F$  in  $\Delta$  as

$$\text{link}_{\Delta} F := \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}.$$

Notice that  $\text{link}_{\Delta} \emptyset = \Delta$ .

**Theorem 2.5.** (Hochster's formula). *Let  $\Delta$  be a simplicial complex on  $[n]$ . Then, the multigraded Betti numbers  $\beta_{i,\underline{a}}(I_{\Delta}^*) \neq 0$  if and only if  $\text{supp}(\underline{a}) = [n] \setminus G$ , for a suitable  $G \in \Delta$ ; in this case  $\beta_{i,\underline{a}}(I_{\Delta}^*) = \tilde{\beta}_{i-1}(\text{link}_{\Delta} G; \mathbb{F})$ . In particular,*

$$\beta_{i,j}(I_{\Delta}^*) = \sum_{G \in \Delta, |G|=n-j} \tilde{\beta}_{i-1}(\text{link}_{\Delta} G; \mathbb{F}).$$

**Remark 2.6.** As mentioned above, Hochster's formula allows to relate explicitly the homology of a simplicial complex  $\Delta$  with the graded Betti numbers of  $I_{\Delta}^*$ . For  $j = n$ , we obtain  $\beta_{i,n}(I_{\Delta}^*) = \tilde{\beta}_{i-1}(\Delta; \mathbb{F})$ .

### 3. BETTI SPLITTING FOR SIMPLICIAL COMPLEXES

Using Alexander dual ideals, it is possible to define the Betti splitting condition for a simplicial complex  $\Delta$ . The main result of this section (Theorem 3.10) describe this condition recursively, in terms of Betti splitting of suitable links. Before proving it, we need some definitions.

**Definition 3.1.** Let  $\Delta$  be a simplicial complex and  $\mathcal{F}(\Delta) = \mathcal{F}_1 \sqcup \mathcal{F}_2$  a partition of  $\mathcal{F}(\Delta)$ . Let  $\Delta_1 = \langle F : F \in \mathcal{F}_1 \rangle$  and  $\Delta_2 = \langle F : F \in \mathcal{F}_2 \rangle$ . We call  $\Delta = \Delta_1 \cup \Delta_2$  a *standard decomposition* of  $\Delta$ .

**Remark 3.2.** The previous definition is the combinatorial counterpart of the assumption on disjoint minimal systems of generators.

Definition 2.1 for squarefree monomial ideals yields the following natural version for simplicial complexes.

**Definition 3.3.** A standard decomposition  $\Delta = \Delta_1 \cup \Delta_2$  of a simplicial complex  $\Delta$  on  $n$  vertices is called a *Betti splitting* of  $\Delta$  over  $\mathbb{F}$  if  $I_{\Delta}^* = I_{\Delta_1}^* + I_{\Delta_2}^*$  is a Betti splitting of  $I_{\Delta}^* \subseteq \mathbb{F}[x_1, \dots, x_n]$ .

**Remark 3.4.** In the previous definition and in the rest of the paper, the Alexander duals ideals are computed with respect to *all* the  $n$  variables of  $\mathbb{F}[x_1, \dots, x_n]$ .

The following key definition is useful to state Theorem 3.10.

**Definition 3.5.** Let  $\Delta = \Delta_1 \cup \Delta_2$  be a standard decomposition of a simplicial complex  $\Delta$ . We say that  $\Delta = \Delta_1 \cup \Delta_2$  is a *homological splitting* of  $\Delta$  over  $\mathbb{F}$  if  $\Delta_1 \cap \Delta_2 = \emptyset$  or if

$$\tilde{\beta}_k(\Delta; \mathbb{F}) = \tilde{\beta}_k(\Delta_1; \mathbb{F}) + \tilde{\beta}_k(\Delta_2; \mathbb{F}) + \tilde{\beta}_{k-1}(\Delta_1 \cap \Delta_2; \mathbb{F}), \text{ for every } k \in \mathbb{N}.$$

**Remark 3.6.** By Theorem 2.5 and Remark 3.2, a homological splitting of  $\Delta$  is equivalent to a Betti splitting of  $I_\Delta^*$  for  $j = n$  and every  $i \in \mathbb{N}$ . In this way, if  $\Delta = \Delta_1 \cup \Delta_2$  is a Betti splitting of  $\Delta$ , then  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$ . In general, the converse does not hold.

In the following example, we show that a homological splitting is not in general a Betti splitting. We will describe a class of simplicial complexes and standard decompositions for which the two notions are equivalent (see Theorem 5.7).

**Example 3.7.** Let  $\Delta$  be the simplicial complex defined by  $\langle 123, 234, 245, 345 \rangle$ . Clearly  $\Delta = \langle 123, 245 \rangle \cup \langle 234, 345 \rangle$  is a standard decomposition of  $\Delta$ . It is easy to see that this is a homological splitting over every field  $\mathbb{F}$ . But it is not a Betti splitting, since  $I_\Delta^*$  is the ideal considered in Example 2.2 and the splitting above is the algebraic counterpart of the splitting here.

For sake of completeness, in the next result we give a straightforward equivalent condition to homological splitting condition.

**Proposition 3.8.** Let  $\Delta$  be a simplicial complex and  $\Delta = \Delta_1 \cup \Delta_2$  be a standard decomposition of  $\Delta$ . Then, the following are equivalent:

1.  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ ;
2. The maps  $\tilde{H}_k(\Delta_1 \cap \Delta_2; \mathbb{F}) \xrightarrow{\phi_k} \tilde{H}_k(\Delta_1; \mathbb{F}) \oplus \tilde{H}_k(\Delta_2; \mathbb{F})$  in the Mayer-Vietoris sequence are zero for every  $k \in \mathbb{N}$ .

*Proof.* Consider the Mayer-Vietoris exact sequence in homology arising from the decomposition  $\Delta = \Delta_1 \cup \Delta_2$  (for sake of simplicity we omit the field  $\mathbb{F}$  in the notations):

$$\dots \rightarrow \tilde{H}_k(\Delta_1 \cap \Delta_2) \xrightarrow{\phi_k} \bigoplus_{i=1}^2 \tilde{H}_k(\Delta_i) \xrightarrow{\psi_k} \tilde{H}_k(\Delta) \xrightarrow{\partial_k} \tilde{H}_{k-1}(\Delta_1 \cap \Delta_2) \xrightarrow{\phi_{k-1}} \bigoplus_{i=1}^2 \tilde{H}_{k-1}(\Delta_i) \rightarrow \dots$$

First we prove that 2. implies 1. If  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$ , we have nothing to prove. Assume  $\Delta_1 \cap \Delta_2 \neq \{\emptyset\}$ , then  $\tilde{H}_{-1}(\Delta_1 \cap \Delta_2) = 0$ . From the long exact sequence above we get

$$\begin{aligned} \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) &= \dim_{\mathbb{F}} \text{Ker}(\psi_k) + \dim_{\mathbb{F}} \text{Im}(\psi_k) = \dim_{\mathbb{F}} \text{Im}(\phi_k) + \dim_{\mathbb{F}} \text{Im}(\psi_k) \\ &= \dim_{\mathbb{F}} \text{Im}(\psi_k) = \dim_{\mathbb{F}} \text{Ker}(\partial_k) = \tilde{\beta}_k(\Delta) - \dim_{\mathbb{F}} \text{Im}(\partial_k) \\ &= \tilde{\beta}_k(\Delta) - \dim_{\mathbb{F}} \text{Ker}(\phi_{k-1}) = \tilde{\beta}_k(\Delta) - \tilde{\beta}_{k-1}(\Delta_1 \cap \Delta_2). \end{aligned}$$

To prove that 1. implies 2. we proceed by induction on  $k \geq 0$ . If  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$ , then all the maps  $\phi_k = 0$ , for  $k \in \mathbb{N}$ . Assume  $\Delta_1 \cap \Delta_2 \neq \{\emptyset\}$ . For  $k = 0$  we have  $\dim_{\mathbb{F}} \text{Im}(\phi_0) =$

$\dim_{\mathbb{F}} \text{Ker}(\phi_0) = \tilde{\beta}_0(\Delta_1) + \tilde{\beta}_0(\Delta_2) - \dim_{\mathbb{F}} \text{Im}(\psi_0) = \tilde{\beta}_0(\Delta_1) + \tilde{\beta}_0(\Delta_2) - \tilde{\beta}_0(\Delta) = 0$ . Assume  $k \geq 1$  and  $\phi_{k-1} = 0$ ; we prove that  $\phi_k = 0$  using an argument similar to the previous one:

$$\begin{aligned}
\dim_{\mathbb{F}} \text{Im}(\phi_k) &= \dim_{\mathbb{F}} \text{Ker}(\psi_k) = \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) - \dim_{\mathbb{F}} \text{Im}(\psi_k) \\
&= \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) - \dim_{\mathbb{F}} \text{Ker}(\partial_k) \\
&= \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) - \tilde{\beta}_k(\Delta) + \dim_{\mathbb{F}} \text{Im}(\partial_k) \\
&= \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) - \tilde{\beta}_k(\Delta) + \dim_{\mathbb{F}} \text{Ker}(\phi_{k-1}) \\
&= \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) - \tilde{\beta}_k(\Delta) + \tilde{\beta}_{k-1}(\Delta_1 \cap \Delta_2) = 0. \quad \square
\end{aligned}$$

**Corollary 3.9.** *Let  $\Delta$  be a simplicial complex and  $\Delta = \Delta_1 \cup \Delta_2$  be a standard decomposition of  $\Delta$ . If  $\Delta_1 \cap \Delta_2$  is acyclic over  $\mathbb{F}$  then  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ .*

The next algorithm checks if a given standard decomposition of a simplicial complex is a homological splitting over a given field  $\mathbb{F}$ . It can be given another version of it using Proposition 3.8.

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**Algorithm 1** *isHomologicalSplitting*( $\Delta, \Delta_1, \Delta_2, \mathbb{F}$ )

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**Input:**  $\mathbb{F}$ , a field  
**Input:**  $\Delta$ , a simplicial complex of dimension  $d$   
**Input:**  $(\Delta_1, \Delta_2)$  a standard decomposition of  $\Delta$   
 $N := 0$   
**if**  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$  **then**  
    **return** TRUE  
**end if**  
Compute the homology of  $\Delta$ ,  $\Delta_1$ ,  $\Delta_2$  and  $\Delta_1 \cap \Delta_2$  with coefficients in  $\mathbb{F}$   
**for**  $0 \leq k \leq \dim(\Delta)$  **do**  
    **if**  $\tilde{\beta}_k(\Delta; \mathbb{F}) = \tilde{\beta}_k(\Delta_1; \mathbb{F}) + \tilde{\beta}_k(\Delta_2; \mathbb{F}) + \tilde{\beta}_{k-1}(\Delta_1 \cap \Delta_2; \mathbb{F})$  **then**  
         $N := N + 1$   
    **end if**  
**end for**  
**if**  $N = \dim(\Delta) + 1$  **then**  
    **return** TRUE  
**else**  
    **return** FALSE  
**end if**

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Finally, we are able to prove the main theorem of this section: a Betti splitting for  $I_{\Delta}^*$  can be characterized in terms of homological splittings or *recursively* in terms of Betti splitting of suitable links.

**Theorem 3.10.** *Let  $\Delta = \Delta_1 \cup \Delta_2$  be a standard decomposition of a simplicial complex  $\Delta$ . Then, the following statements are equivalent:*

1.  $\Delta = \Delta_1 \cup \Delta_2$  is a Betti splitting of  $\Delta$  over  $\mathbb{F}$ ;

2.  $\text{link}_\Delta F = \text{link}_{\Delta_1} F \cup \text{link}_{\Delta_2} F$  is a homological splitting of  $\text{link}_\Delta F$  over  $\mathbb{F}$ , for each face  $F \in \Delta_1 \cap \Delta_2$ ;
3.  $\text{link}_\Delta v = \text{link}_{\Delta_1} v \cup \text{link}_{\Delta_2} v$  is a Betti splitting over  $\mathbb{F}$  of  $\text{link}_\Delta v$ , for each vertex  $v \in \Delta_1 \cap \Delta_2$  and  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ .

*Proof.* Notice that since  $\Delta = \Delta_1 \cup \Delta_2$  is a standard decomposition of  $\Delta$ , then  $\text{link}_\Delta F = \text{link}_{\Delta_1} F \cup \text{link}_{\Delta_2} F$  is a standard decomposition of  $\text{link}_\Delta F$ , for every  $F \in \Delta$ .

First we prove that 1. implies 2. If  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$  we have nothing to prove, since  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ . We may assume  $\Delta_1 \cap \Delta_2 \neq \{\emptyset\}$ . By definition,

$$\beta_{i,j}(I_\Delta^*) = \beta_{i,j}(I_{\Delta_1}^*) + \beta_{i,j}(I_{\Delta_2}^*) + \beta_{i-1,j}(I_{\Delta_1}^* \cap I_{\Delta_2}^*), \text{ for every } i, j \in \mathbb{N}.$$

Since all the maps  $\text{Tor}_i(I_{\Delta_1}^* \cap I_{\Delta_2}^*, \mathbb{F}) \rightarrow \text{Tor}_i(I_{\Delta_1}^*, \mathbb{F}) \oplus \text{Tor}_i(I_{\Delta_2}^*, \mathbb{F})$  are trivial, then all the maps  $\text{Tor}_i(I_{\Delta_1}^* \cap I_{\Delta_2}^*, \mathbb{F})_{\underline{a}} \rightarrow \text{Tor}_i(I_{\Delta_1}^*, \mathbb{F})_{\underline{a}} \oplus \text{Tor}_i(I_{\Delta_2}^*, \mathbb{F})_{\underline{a}}$  are trivial, for every multidegree  $\underline{a} \in \mathbb{Z}^n$ . Then,

$$\beta_{i,\underline{a}}(I_\Delta^*) = \beta_{i,\underline{a}}(I_{\Delta_1}^*) + \beta_{i,\underline{a}}(I_{\Delta_2}^*) + \beta_{i-1,\underline{a}}(I_{\Delta_1}^* \cap I_{\Delta_2}^*), \text{ for every } i \in \mathbb{N} \text{ and } \underline{a} \in \mathbb{Z}^n.$$

Recall that  $I_{\Delta_1}^* \cap I_{\Delta_2}^* = I_{\Delta_1 \cap \Delta_2}^*$ . By Hochster's formula,  $\beta_{i,\underline{a}}(I_\Delta^*) \neq 0$  if and only if  $\text{supp}(\underline{a}) = [n] \setminus F$ , for a suitable face  $F \in \Delta$ ; in this case  $\beta_{i,\underline{a}}(I_\Delta^*) = \tilde{\beta}_{i-1}(\text{link}_\Delta F)$ , for every  $i \in \mathbb{N}$ . Let  $F \in \Delta_1 \cap \Delta_2$ . Then,

$$\tilde{\beta}_{i-1}(\text{link}_\Delta F; \mathbb{F}) = \tilde{\beta}_{i-1}(\text{link}_{\Delta_1} F; \mathbb{F}) + \tilde{\beta}_{i-1}(\text{link}_{\Delta_2} F; \mathbb{F}) + \tilde{\beta}_{i-2}(\text{link}_{\Delta_1 \cap \Delta_2} F; \mathbb{F}),$$

for every  $i \in \mathbb{N}$ .

Now we prove 2. implies 3. Consider  $F = \emptyset$ . By assumption, we have that  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ . If  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$ , we are done. Let  $v \in \Delta_1 \cap \Delta_2$  be a vertex and  $\underline{a} \in \mathbb{Z}^n$  such that  $\text{supp}(\underline{a}) = [n] \setminus G$  for a suitable face  $G \in \text{link}_\Delta v$ . If  $G \notin \text{link}_{\Delta_1} v$  (the same for  $\text{link}_{\Delta_2} v$ ), it follows that  $\text{link}_\Delta v = \text{link}_{\Delta_2} v$  and we have nothing to prove. We may assume  $G \in \text{link}_{\Delta_1} v \cap \text{link}_{\Delta_2} v = \text{link}_{\Delta_1 \cap \Delta_2} v$ . By definition, we have that  $F = G \cup \{v\}$  is a face of  $\Delta_1 \cap \Delta_2$ . By 2., we know that  $\text{link}_\Delta F = \text{link}_{\Delta_1} F \cup \text{link}_{\Delta_2} F$  is a homological splitting of  $\text{link}_\Delta F$ . Note that  $\tilde{\beta}_{i-1}(\text{link}_\Delta F) = \tilde{\beta}_{i-1}(\text{link}_{\text{link}_\Delta v} G) = \beta_{i,\underline{a}}(I_{\text{link}_\Delta v}^*)$ , over  $\mathbb{F}$ . Then,

$$\beta_{i,\underline{a}}(I_{\text{link}_\Delta v}^*) = \beta_{i,\underline{a}}(I_{\text{link}_{\Delta_1} v}^*) + \beta_{i,\underline{a}}(I_{\text{link}_{\Delta_2} v}^*) + \beta_{i-1,\underline{a}}(I_{\text{link}_{\Delta_1 \cap \Delta_2} v}^*).$$

Summing over all  $\underline{a} \in \mathbb{Z}^n$  such that  $|\text{supp}(\underline{a})| = j$ , we obtain the desired splitting formula for  $j$  and for every  $i \in \mathbb{N}$ .

Finally, we prove 3. implies 1. Since  $\Delta = \Delta_1 \cup \Delta_2$  is a homological splitting of  $\Delta$ , for  $j = n$  the splitting formula holds. Let  $j \neq n$  and  $\underline{a} \in \mathbb{Z}^n$ , such that  $\text{supp}(\underline{a}) = [n] \setminus F$ , for some face  $F \in \Delta$  with  $|F| = n - j$ . If  $F \in \Delta \setminus \Delta_1$  (the same for  $\Delta_2$ ) we have  $\text{link}_\Delta F = \text{link}_{\Delta_2} F$ . Then,  $\beta_{i,\underline{a}}(I_\Delta^*) = \beta_{i,\underline{a}}(I_{\Delta_2}^*)$  and  $\beta_{i,\underline{a}}(I_{\Delta_1}^*) = \beta_{i-1,\underline{a}}(I_{\Delta_1}^* \cap I_{\Delta_2}^*) = 0$ . Then, we may assume  $F \in \Delta_1 \cap \Delta_2$ . Let  $v \in F$  and  $\underline{b} \in \mathbb{Z}^n$  such that  $\text{supp}(\underline{b}) = [n] \setminus (F \setminus v)$ . By assumption,

$$\beta_{i,\underline{b}}(I_{\text{link}_\Delta v}^*) = \beta_{i,\underline{b}}(I_{\text{link}_{\Delta_1} v}^*) + \beta_{i,\underline{b}}(I_{\text{link}_{\Delta_2} v}^*) + \beta_{i-1,\underline{b}}(I_{\text{link}_{\Delta_1} v}^* \cap I_{\text{link}_{\Delta_2} v}^*),$$

for every  $i \in \mathbb{N}$ . By Hochster's formula,

$$\beta_{i,\underline{b}}(I_{\text{link}_{\Delta} v}^*) = \tilde{\beta}_{i-1}(\text{link}_{\text{link}_{\Delta} v}(F \setminus \{v\})) = \tilde{\beta}_{i-1}(\text{link}_{\Delta} F) = \beta_{i,\underline{a}}(I_{\Delta}^*).$$

Applying the previous argument to  $\Delta_1$ ,  $\Delta_2$  and  $\Delta_1 \cap \Delta_2$ , we get

$$\beta_{i,\underline{a}}(I_{\Delta}^*) = \beta_{i,\underline{a}}(I_{\Delta_1}^*) + \beta_{i,\underline{a}}(I_{\Delta_2}^*) + \beta_{i-1,\underline{a}}(I_{\Delta_1}^* \cap I_{\Delta_2}^*).$$

Summing over all  $\underline{a} \in \mathbb{Z}^n$  such that  $|\text{supp}(\underline{a})| = j$  we conclude the proof.  $\square$

Using Theorem 3.10 we are able to present the following algorithm: it checks if a given standard decomposition of a simplicial complex is a Betti splitting over a given field  $\mathbb{F}$ . It is possible to give another formulation of the algorithm taking advantage of the third condition of the theorem.

---

**Algorithm 2** *IsBettiSplitting*( $\Delta, \Delta_1, \Delta_2, \mathbb{F}$ )

---

**Input:**  $\mathbb{F}$ , a field  
**Input:**  $\Delta$ , a simplicial complex of dimension  $d$   
**Input:**  $\Delta = \Delta_1 \cup \Delta_2$ , a standard decomposition of  $\Delta$   
 $N := 0$   
**for each** face  $F$  of  $\Delta_1 \cap \Delta_2$  **do**  
    Compute  $\text{link}_{\Delta} F$ ,  $\text{link}_{\Delta_1} F$  and  $\text{link}_{\Delta_2} F$   
    **if not** *isHomologicalSplitting*( $\text{link}_{\Delta} F, \text{link}_{\Delta_1} F, \text{link}_{\Delta_2} F, \mathbb{F}$ ) **then**  
         $N := N + 1$  **break**  
    **end if**  
**end for**  
**if**  $N > 0$  **then**  
    **return** FALSE  
**else**  
    **return** TRUE  
**end if**

---

#### 4. ESSENTIAL FACETS

In several cases, a nice Betti splitting for a simplicial complex  $\Delta$  is given by a standard decomposition obtained removing a single facet from  $\Delta$ . By Theorem 3.10(3.), we start studying situations for which the standard decomposition  $\Delta = \langle G \in \mathcal{F}(\Delta) : G \neq F \rangle \cup \langle F \rangle$  is a homological splitting for  $\Delta$ .

The following result is very useful for our purpose.

**Proposition 4.1.** ([4], Section 3). *Let  $\Delta$  be a simplicial complex,  $F \in \mathcal{F}(\Delta)$  be a facet of  $\Delta$  of dimension  $d$  and  $\mathbb{F}$  be a field. Setting  $\Delta' = \Delta \setminus \{F\}$ , we have that:*

- $F$  belongs to a  $d$ -cycle of  $\Delta$  if and only if

$$\tilde{\beta}_k(\Delta'; \mathbb{F}) = \begin{cases} \tilde{\beta}_k(\Delta; \mathbb{F}) - 1 & \text{if } k = d \\ \tilde{\beta}_k(\Delta; \mathbb{F}) & \text{otherwise} \end{cases}$$



- $F$  does not belong to a  $d$ -cycle of  $\Delta$  if and only if

$$\tilde{\beta}_k(\Delta'; \mathbb{F}) = \begin{cases} \tilde{\beta}_k(\Delta; \mathbb{F}) + 1 & \text{if } k = d - 1 \\ \tilde{\beta}_k(\Delta; \mathbb{F}) & \text{otherwise} \end{cases}$$

**Remark 4.2.** Proposition 4.1 ensures that removing from  $\Delta$  a facet of dimension  $d$  only affects either the  $d$ -th or the  $(d - 1)$ -th homology group of  $\Delta$  and that only one of these situations may occur.

Given a simplicial complex  $\Delta$ , we call *essential* each facet of dimension  $d$  of  $\Delta$  whose removal only affects the  $d^{\text{th}}$  homology group of  $\Delta$ .

**Definition 4.3.** Let  $\Delta$  be a simplicial complex and let  $F \in \mathcal{F}(\Delta)$  be a facet of dimension  $d$  in  $\Delta$ . We call  $F$  an *essential facet* over a field  $\mathbb{F}$  if and only if

$$\tilde{\beta}_k(\Delta \setminus \{F\}; \mathbb{F}) = \begin{cases} \tilde{\beta}_k(\Delta; \mathbb{F}) - 1 & \text{if } k = d \\ \tilde{\beta}_k(\Delta; \mathbb{F}) & \text{otherwise} \end{cases}$$

The next lemma shows that an essential facet induces a standard decomposition of  $\Delta$ .

**Lemma 4.4.** Let  $\Delta$  be a simplicial complex and let  $F \in \mathcal{F}(\Delta)$  be a  $d$ -dimensional essential facet over  $\mathbb{F}$ . Then,  $\langle G \mid G \in \mathcal{F}(\Delta), G \neq F \rangle = \Delta \setminus \{F\}$ .

*Proof.* The inclusion " $\subseteq$ " is trivial. Conversely, it suffices to show that for every  $(d - 1)$ -face  $H \subseteq F$  we have  $H \in \langle G \mid G \in \mathcal{F}(\Delta), G \neq F \rangle$ . Since the facet  $F$  is essential,  $F$  belongs to a  $d$ -cycle in  $\Delta$  and so  $\partial F$  is a boundary in  $\Delta \setminus \{F\}$ , where  $\partial$  denotes the boundary map in the chain complex of  $\Delta$ . Then, since  $H \in \partial F$ , there exists a facet  $G \in \mathcal{F}(\Delta)$ ,  $G \neq F$ , such that  $H \subseteq G$ .  $\square$

Finally, in the next result, we prove that an essential face over  $\mathbb{F}$  induces a homological splitting over  $\mathbb{F}$ .

**Theorem 4.5.** Let  $\Delta$  be a simplicial complex and let  $\mathbb{F}$  be a field. If there exists a  $d$ -dimensional essential facet  $F \in \mathcal{F}(\Delta)$  over  $\mathbb{F}$ , then  $\Delta = \langle G \in \mathcal{F}(\Delta) : G \neq F \rangle \cup \langle F \rangle$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ . Moreover  $\tilde{\beta}_d(\Delta; \mathbb{F}) \neq 0$ .

*Proof.* For simplicity, in the proof we omit the field  $\mathbb{F}$  from the notations. Let  $F \in \mathcal{F}(\Delta)$  be a  $d$ -dimensional essential facet of  $\Delta$  with respect to  $\mathbb{F}$ . Consider the complexes

- $\Delta_1 := \langle G \mid G \in \mathcal{F}(\Delta), G \neq F \rangle$ ,
- $\Delta_2 := \langle F \rangle$ .

By Lemma 4.4,  $\Delta_1 = \Delta \setminus \{F\}$  and  $\Delta = \Delta_1 \cup \Delta_2$  is a standard decomposition of  $\Delta$ . Since  $F$  is a  $d$ -dimensional essential facet, we have that

$$\tilde{\beta}_k(\Delta_1) = \begin{cases} \tilde{\beta}_k(\Delta) - 1 & \text{if } k = d \\ \tilde{\beta}_k(\Delta) & \text{otherwise} \end{cases}$$

Moreover,  $\tilde{\beta}_k(\Delta_2) = 0$ , for  $k \in \mathbb{N}$ .

If  $d = 0$ , we have  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$  and we have nothing to prove. Assume  $d \geq 1$ . Then,  $\Delta_1 \cap \Delta_2$  is homeomorphic to the  $(d - 1)$ -sphere  $\mathbb{S}^{d-1}$ . Therefore, we have  $\tilde{\beta}_{d-1}(\Delta_1 \cap \Delta_2) = 1$

and  $\tilde{\beta}_k(\Delta_1 \cap \Delta_2) = 0$  for  $k \neq d-1$ . It is easy to check that the above Betti numbers satisfy the equations described in Definition 3.5. In fact,

$$\begin{aligned} \tilde{\beta}_k(\Delta_1) + \tilde{\beta}_k(\Delta_2) + \tilde{\beta}_{k-1}(\Delta_1 \cap \Delta_2) &= \tilde{\beta}_k(\Delta) + 0 + 0 = \tilde{\beta}_k(\Delta) \text{ if } k \neq d \text{ and} \\ \tilde{\beta}_d(\Delta_1) + \tilde{\beta}_d(\Delta_2) + \tilde{\beta}_{d-1}(\Delta_1 \cap \Delta_2) &= \tilde{\beta}_d(\Delta) - 1 + 0 + 1 = \tilde{\beta}_d(\Delta). \end{aligned}$$

The last claim is trivial. If  $d = 0$ , the facet  $F$  is an isolated point. If  $d \geq 1$ , clearly  $\tilde{\beta}_d(\Delta) \neq 0$  since  $\tilde{\beta}_{d-1}(\Delta_1 \cap \Delta_2) = 1$ .  $\square$

**Remark 4.6.** We remark that in the previous result, the dimension  $d$  of the essential facet  $F$  is *not* forced to be equal to the dimension of the simplicial complex  $\Delta$ .

In the next example, we show that in general the converse of Theorem 4.5 does not hold, i.e. the condition  $\beta_d(\Delta; \mathbb{F}) \neq 0$  does not imply the existence of essential facet of dimension  $d$ , if  $d \neq \dim(\Delta)$ .

**Example 4.7.** Consider the simplicial complex  $\Delta = \langle 123, 345, 246 \rangle$ . It is immediate to see that  $\tilde{\beta}_1(\Delta; \mathbb{F}) \neq 0$ , for every field  $\mathbb{F}$ . But there is no facet of dimension 1.

If the dimension of the essential facet equals the dimension of  $\Delta$ , a stronger statement holds.

**Theorem 4.8.** *Let  $\Delta$  be a simplicial complex of dimension  $d$  and let  $\mathbb{F}$  be a field. Then, the following conditions are equivalent:*

1.  $\tilde{\beta}_d(\Delta; \mathbb{F}) \neq 0$ ;
2. *there exists a  $d$ -dimensional essential facet  $F \in \mathcal{F}(\Delta)$  over  $\mathbb{F}$ .*

*In this case, there exists a facet  $F \in \mathcal{F}(\Delta)$  such that  $\Delta = \langle G \in \mathcal{F}(\Delta) : G \neq F \rangle \cup \langle F \rangle$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ .*

*Proof.* The implication from 2. to 1. is given by Theorem 4.5. Then, we have to prove only that 1. implies 2. Let  $(\Delta_i)_{i=0, \dots, M}$  be a collection of simplicial complexes such that:

- $\Delta_0 := \{\emptyset\}$ ,  $\Delta_M := \Delta$ ;
- for each  $i \in \{0, \dots, M-1\}$ ,  $\Delta_{i+1} = \Delta_i \cup \{F_{i+1}\}$ , where  $F_{i+1}$  is a single face of  $\Delta$ .

Given such a collection, let  $j := \min\{i \in \{0, \dots, M\} : \beta_d(\Delta_i; \mathbb{F}) \neq 0\}$ . We set  $F := F_j$ , the face introduced in  $\Delta_j$ . Since  $\beta_d(\Delta_j; \mathbb{F}) \neq 0$ ,  $F$  is a facet of dimension  $d$  in  $\Delta$  which belongs to a  $d$ -cycle of  $\Delta_j$ . Then, by Proposition 4.1,  $F$  is an essential facet of  $\Delta$  over  $\mathbb{F}$ .

The last claim is an immediate consequence of Theorem 4.5.  $\square$

**Remark 4.9.** Let  $\Delta$  be a  $d$ -dimensional complex. Clearly in general the existence of a homological splitting  $\Delta = \langle G \in \mathcal{F}(\Delta) : G \neq F \rangle \cup \langle F \rangle$  over  $\mathbb{F}$  induced by a  $d$ -dimensional facet  $F \in \mathcal{F}(\Delta)$  does not imply  $\tilde{\beta}_d(\Delta; \mathbb{F}) \neq 0$  nor that the considered facet is essential. Consider for instance the decomposition  $\langle 123, 345 \rangle \cup \langle 246 \rangle$  of the complex in Example 4.7. It is a homological splitting over every field  $\mathbb{F}$ , by Corollary 3.9, but  $\beta_2(\Delta; \mathbb{F}) = 0$ .

**Remark 4.10.** It is worth of mention that the homological splitting in Theorem 4.8 is ensured for *every* facet of the  $d$ -cycle considered.

The following corollary is an immediate consequence of Theorem 4.8.

**Corollary 4.11.** *Let  $\Delta$  be a simplicial complex of dimension  $d$ . If  $\beta_d(\Delta; \mathbb{F}) \neq 0$ , then  $\Delta$  admits a homological splitting over  $\mathbb{F}$ .*

## 5. BETTI SPLITTING OF MANIFOLDS

In this section, we apply our results to triangulations of manifolds, a large class of simplicial complexes with nice properties. Let us recall briefly some basic definitions.

**Definition 5.1.** A *topological  $d$ -manifold*  $M$  is a Hausdorff space such that every point  $x \in X$  has a neighbourhood which is homeomorphic to the  $d$ -dimensional Euclidean space.

All the manifolds that we consider are closed, i.e. compact and without boundary. A *triangulation* of a manifold  $M$  is a simplicial complex  $\Delta$  such that  $|\Delta| \cong M$ . In this case, from now on we say that  $\Delta$  itself is a  $d$ -manifold. We summarize some properties of a  $d$ -manifold  $\Delta$ :

- $\Delta$  is *pure* and *strongly connected* (i.e. all the facets have the same dimension  $d$  and for every two facets  $F, G$  there exists a path  $H_0 = F, H_1, \dots, H_k, H_{k+1} = G$ , where  $H_i$  are facets such that  $H_i \cap H_{i+1}$  is a  $(d-1)$ -dimensional face of  $\Delta$ , for every  $1 \leq i \leq k$ );
- $\Delta$  is a *pseudomanifold* (i.e. every face of dimension  $d-1$  lies in exactly two facets of  $\Delta$ );
- $\text{link}_\Delta v$  is a  $(d-1)$ -dimensional *homology sphere* (i.e. a topological manifold with the same homology of the sphere, for every field  $\mathbb{F}$ ).

The graded Betti numbers of  $I_\Delta^*$  are essentially known by Hochster's formula, see Remark 5.2. For the case of a manifold  $\Delta$ , it is interesting to investigate if nice topological properties on  $\Delta$  induce nice splitting properties on  $I_\Delta^*$  and how the topology of  $\Delta$  may be reflected on the structure of the resolution of  $I_\Delta^*$ .

**Remark 5.2.** Notice that if  $\Delta$  is a  $d$ -manifold, then  $\beta_i(I_\Delta^*) = f_{d-i}(\Delta) + \tilde{\beta}_{i-1}(\Delta; \mathbb{F})$ , for  $0 \leq i \leq d+1$ , where  $(f_{-1}(\Delta), f_0(\Delta), \dots, f_d(\Delta))$  is the  $f$ -vector of  $\Delta$  and  $f_i(\Delta)$  denotes the number of  $i$ -dimensional faces of  $\Delta$ . In this case,  $I_\Delta^*$  is generated in degree  $n-d-1$ . The equality above follows immediately from Hochster's formula, since  $\text{link}_\Delta(F)$  is a  $(j-1)$ -dimensional homology sphere if  $F \neq \emptyset$  is a face of  $\Delta$  of cardinality  $n-j$ .

A  $d$ -manifold  $M$  is called *orientable* if it has a global consistent choice of orientation. For the formal definition and other details on orientability, we refer to [11, Chapter 3.3]. Examples of orientable manifolds are (homology) spheres, see [11, Corollary 3.28]. The Möbius strip and the Klein bottle are examples of non-orientable spaces.

Denote by  $H_k(\Delta; \mathbb{F})$  the  $k^{\text{th}}$  *non-reduced* homology group of  $\Delta$  with coefficients in  $\mathbb{F}$  and denote by  $\beta_k(\Delta; \mathbb{F})$  its dimension. Recall that  $\beta_k(\Delta; \mathbb{F}) = \tilde{\beta}_k(\Delta; \mathbb{F})$  for  $k \neq 0$  and  $\beta_0(\Delta; \mathbb{F}) = \tilde{\beta}_0(\Delta; \mathbb{F}) + 1$ .

Poincaré duality Theorem is a fundamental result on the topology of orientable  $d$ -manifold. We state it in a classical fashion (for a modern treatment see [11, Theorem 3.30]).

**Theorem 5.3.** (Poincaré duality Theorem). *Let  $\Delta$  be an orientable  $d$ -manifold. Then, for any integer  $k$  and any field  $\mathbb{F}$ ,*

$$H_k(\Delta; \mathbb{F}) \cong H_{d-k}(\Delta; \mathbb{F})$$

**Remark 5.4.** For every  $d$ -manifold, the isomorphism of Theorem 5.3 holds for  $\mathbb{F} = \mathbb{Z}_2$ .

Using Poincaré duality, we prove that the facets of an orientable manifold are all essential.

**Proposition 5.5.** *Let  $\mathbb{F}$  be a field and  $\Delta$  be an orientable manifold. Then, any facet  $F \in \mathcal{F}(\Delta)$  is an essential facet over  $\mathbb{F}$ . Moreover  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a homological splitting over  $\mathbb{F}$ .*

*Proof.* Poincaré duality Theorem 5.3 ensures that  $H_d(\Delta; \mathbb{F}) \cong H_0(\Delta; \mathbb{F}) \neq 0$ . Furthermore, the geometrical realizations of the  $d$ -cycles of  $H_d(\Delta; \mathbb{F})$  coincide with the connected components of  $\Delta$  itself. So, given any facet  $F$  in  $\Delta$ ,  $F$  belongs to a  $d$ -cycle of  $\Delta$ . Then, by Theorem 4.8 and Remark 4.10,  $F$  is an essential facet of  $\Delta$  over  $\mathbb{F}$ .  $\square$

By Remark 5.4, we can state the following result, the proof of which is similar to the previous one.

**Proposition 5.6.** *Let  $\Delta$  be a manifold. Then, any facet  $F \in \mathcal{F}(\Delta)$  is an essential facet over  $\mathbb{Z}_2$ . Moreover  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a homological splitting over  $\mathbb{Z}_2$ .*

For manifolds, the Betti splitting condition is equivalent to the homological splitting condition.

**Theorem 5.7.** *Let  $\Delta$  be a  $d$ -manifold and let  $\mathbb{F}$  be a field. Given a facet  $F \in \mathcal{F}(\Delta)$ , the following statements are equivalent:*

1.  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a homological splitting of  $\Delta$  over  $\mathbb{F}$ ;
2.  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a Betti splitting of  $\Delta$  over  $\mathbb{F}$ .

*Proof.* Since a Betti splitting is always a homological splitting, it is enough to prove that 1. implies 2. Consider  $\Delta_1 = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle$  and  $\Delta_2 = \langle F \rangle$ . We prove the claim by induction on the dimension  $d$  of the manifold  $\Delta$ . For  $d = 0$ ,  $\Delta$  is a set of isolated points. In this case,  $\Delta_1 \cap \Delta_2 = \{\emptyset\}$ , then we have nothing to prove. Assume the implication proved up to dimension  $k$  and we prove it for dimension  $d = k + 1$ . Let  $v$  be a vertex of  $\Delta_1 \cap \Delta_2$ . Recall that  $\text{link}_\Delta v$  is a  $k$ -dimensional homology sphere. Notice that  $\text{link}_{\Delta_2} v$  is a facet of  $\text{link}_\Delta v$ . By Proposition 5.5,  $\text{link}_\Delta v = \text{link}_{\Delta_1} v \cup \text{link}_{\Delta_2} v$  is a homological splitting of  $\text{link}_\Delta v$  over  $\mathbb{F}$ , since any homology sphere is an orientable manifold. By induction, it is also a Betti splitting. Then, by Theorem 3.10, we have that the homological splitting  $\Delta = \Delta_1 \cup \Delta_2$  is a Betti splitting of  $\Delta$ .  $\square$

By Proposition 5.5 and Proposition 5.6 respectively, the following results are immediate consequences of Theorem 5.7.

**Corollary 5.8.** *Let  $\mathbb{F}$  be a field and  $\Delta$  be an orientable manifold. Then, for each facet  $F \in \mathcal{F}(\Delta)$ , the standard decomposition  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a Betti splitting over  $\mathbb{F}$ .*

**Corollary 5.9.** *Let  $\Delta$  be a manifold. Then, for each facet  $F \in \mathcal{F}(\Delta)$ , the standard decomposition  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a Betti splitting over  $\mathbb{Z}_2$ .*

In the following example, we show that Corollary 5.8 does not hold for manifolds with boundary.

**Example 5.10.** It can be proved that the standard triangulation of Rudin's ball does not admit a Betti splitting obtained removing a single facet; it admits, anyway, another kind of splitting (see [1, Example 4.7]).

**Remark 5.11.** Let  $d \geq 1$ . For a  $d$ -manifold, the following are equivalent:

1.  $\Delta$  is orientable;
2.  $\beta_d(\Delta; \mathbb{F}) \neq 0$  for every field  $\mathbb{F}$ ;
3. for every facet  $F \in \mathcal{F}(\Delta)$  we have that  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a Betti splitting over every field  $\mathbb{F}$ ;
4. there exists one facet  $F \in \mathcal{F}(\Delta)$  such that  $\Delta = \langle G | G \in \mathcal{F}(\Delta), G \neq F \rangle \cup \langle F \rangle$  is a Betti splitting over every field  $\mathbb{F}$ .

1. is equivalent to 2., since if  $\Delta$  is orientable, then  $\tilde{H}_d(\Delta; \mathbb{F}) \neq 0$  for every field  $\mathbb{F}$  and the  $d$ -cycles are given by the connected components of the manifold itself; moreover, if  $\Delta$  is not orientable, then  $\tilde{H}_d(\Delta; \mathbb{F}) = 0$  for every field  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) \neq 2$ . 2. implies 3. by Proposition 5.5 and Theorem 5.7. Clearly, 3. implies 4. Finally, 4. implies 2., by the last part of Theorem 4.5.

## 6. APPLICATIONS

The results of the previous sections describe a large class of discretized topological spaces admitting a Betti splitting. In this section, we present several interesting examples.

Corollary 5.8 ensures us that *every* triangulation of an orientable manifold, for instance an  $n$ -sphere  $\mathbb{S}^n$ , a torus  $T$  and a projective plane of odd dimension  $\mathbb{RP}^{2n+1}$ , admits a Betti splitting decomposition over every field  $\mathbb{F}$ , induced by the removal of any of its top dimensional simplices. Another interesting example is provided by the triangulation  $\Delta$  given in [14] of the lens space  $L(3, 1)$  (for more details see Example 6.3).

Theorem 4.5 ensures the existence of a homological splitting also for non-manifold simplicial complexes. For instance, consider the mod 3 Moore space  $M$  [3] depicted in Figure 1(c).  $M$  is a 2-dimensional simplicial complex with  $\beta_2(M; \mathbb{Z}_3) \neq 0$ . So, by Corollary 4.11, it admits a homological splitting over  $\mathbb{Z}_3$ . In this case, all the facets of  $M$  induce a homological splitting; it can be easily proved that it is also a Betti splitting. The situation is completely different if  $\text{char}(\mathbb{F}) \neq 3$ .

In view of Remark 5.11, we know that every triangulation of some relevant non-orientable manifolds, such as a projective space of even dimension  $\mathbb{RP}^{2n}$  and the Klein bottle  $K$  do not admit Betti splitting or homological splitting when we remove a single facet if  $\text{char}(\mathbb{F}) \neq 2$ . Indeed, we are able to show that suitable triangulations of these spaces (see Figure 1(a) and Figure 1(b)) do not admit *any* Betti splitting over a field of characteristic different from 2. For the dunce hat  $D$ , see Figure 1(d) the pathology is the same, but over every field. The first author already proved in [1] that the given triangulation of the dunce hat does not admit Betti splitting.

The spaces considered and their topological properties are summarized in Table 1:

The notion we are going to introduce allows us to detect a large class of simplicial complexes that do not admit Betti splitting, i.e. every possible standard decomposition is not a Betti splitting.

TABLE 1. Relevant properties of the considered simplicial complexes.

| $\Delta$             | Manifold | Orientable | $\mathbb{F}$ s.t. $\tilde{\beta}_d(\Delta; \mathbb{F}) \neq 0$ |
|----------------------|----------|------------|--|
| $\mathbb{RP}^{2n+1}$ | ✓        | ✓          | any $\mathbb{F}$   |
| $L(3, 1)$            | ✓        | ✓          | any $\mathbb{F}$   |
| $\mathbb{RP}^{2n}$   | ✓        | ✗          | $\mathbb{F}$ with $\text{char}(\mathbb{F}) = 2$                |
| $K$                  | ✓        | ✗          | $\mathbb{F}$ with $\text{char}(\mathbb{F}) = 2$                |
| $M$                  | ✗        | —          | $\mathbb{F}$ with $\text{char}(\mathbb{F}) = 3$                |
| $D$                  | ✗        | —          | none   |

**Definition 6.1.** Let  $\mathbb{F}$  be a field and let  $\Delta$  be a simplicial complex of dimension  $d$ .  $\Delta$  is called *non-trivially decomposable* over  $\mathbb{F}$  if, for each standard decomposition  $\Delta_1 \cup \Delta_2$  of  $\Delta$ ,  $\tilde{\beta}_{d-1}(\Delta_1 \cap \Delta_2; \mathbb{F}) \neq 0$ .

As an immediate consequence of Definition 3.5, we can state the following proposition.

**Proposition 6.2.** *Let  $\mathbb{F}$  be a field and let  $\Delta$  be a simplicial complex of dimension  $d > 1$  with  $\tilde{\beta}_d(\Delta; \mathbb{F}) = 0$ . If  $\Delta$  is non-trivially decomposable over  $\mathbb{F}$ , then  $\Delta$  does not admit homological splitting, and so, Betti splitting.*

Thanks to the proposition above, given a simplicial complex  $\Delta$  of dimension  $d > 1$  with  $\tilde{\beta}_d(\Delta; \mathbb{F}) = 0$  and a field  $\mathbb{F}$ , to check if  $\Delta$  is non-trivially decomposable is enough to ensure that  $\Delta$  does not admit any homological and Betti splitting. The property of being non-trivially decomposable can be algorithmically checked by performing the following algorithm.

---

**Algorithm 3** *isNonTriviallyDecomposable*( $\Delta, \mathbb{F}$ )

---

**Input:**  $\mathbb{F}$ , a field  
**Input:**  $\Delta$ , a simplicial complex of dimension  $d$   
 Compute all the possible standard decomposition  $\Delta_1 \cup \Delta_2$  of  $\Delta$   
**for each** standard decomposition  $\Delta_1 \cup \Delta_2$  of  $\Delta$  **do**  
   Compute  $\tilde{\beta}_{d-1}(\Delta_1 \cap \Delta_2; \mathbb{F})$   
   **if**  $\tilde{\beta}_{d-1}(\Delta_1 \cap \Delta_2; \mathbb{F}) = 0$  **then**  
     **return** FALSE  
   **end if**  
**end for**  
**return** TRUE

---

Algorithm 3 retrieves all the  $\frac{1}{2} \sum_{k=1}^{|\mathcal{F}(\Delta)|-1} \binom{|\mathcal{F}(\Delta)|}{k} = 2^{|\mathcal{F}(\Delta)|-1} - 1$  possible standard decompositions of  $\Delta$ . For each decomposition  $\Delta_1 \cup \Delta_2$  of  $\Delta$ , Algorithm 3 computes the  $(d-1)^{\text{th}}$  homology group of  $\Delta_1 \cap \Delta_2$  over  $\mathbb{F}$ . The complex  $\Delta$  is non-trivially decomposable if only if all the computed homology groups are not trivial.

A version of Algorithm 3 for 2-dimensional simplicial complexes has been developed and implemented in Python. The source code of this tool and of the other algorithms described

in the paper can be found in <https://github.com/fugacci/Check-Decompositions.git>. In this algorithm, we take advantage of the fact that  $\Delta_1 \cap \Delta_2$  has dimension 1, i.e. it is a graph.

Using Proposition 6.2 and Algorithm 3, we prove that a homological splitting, and so a Betti splitting, is not available for several simplicial complexes  $\Delta$ , considering fields  $\mathbb{F}$  for which  $\tilde{\beta}_2(\Delta; \mathbb{F}) = 0$  and proving that they are not trivially-decomposable. The considered spaces are depicted in Figure 1: the real projective plane  $\mathbb{RP}^2$ , the Klein bottle  $K$ , the mod 3 Moore space  $M$  and the Duncce hat  $D$ .

Table 2 focus on simplicial complexes which have been detected as non-trivially decomposable. For each simplicial complex  $\Delta$ ,  $n_V$  denotes the number of vertices of the chosen triangulation. In the third column, the number of standard decompositions that have to be checked is showed, while the last column shows the required time (in seconds) to perform the entire computation and to check if  $\Delta$  is non-trivially decomposable.

TABLE 2. Statistics of the computation.

| $\Delta$        | $n_V$ | $ \mathcal{F}(\Delta) $ | Number of decompositions | Time in seconds |
|-----------------|-------|-------------------------|--------------------------|-----------------|
| $\mathbb{RP}^2$ | 6     | 10                      | 511                      | 0.16            |
| $K$             | 8     | 16                      | 32767                    | 16.78           |
| $D$             | 8     | 17                      | 65535                    | 31.67           |
| $M$             | 9     | 19                      | 262143                   | 196.13          |

**Example 6.3.** Consider the triangulation  $\Delta$  given in [14] of the lens space  $L(3, 1)$  (for more details on this space see [11]). It is an orientable 3-manifold. Then, by Corollary 5.8 it admits Betti splitting induced by the removal of any of its facets. The Alexander dual ideal  $I_\Delta^*$  is the first example in literature of an ideal with characteristic dependent resolution admitting Betti splitting over every field.

Proposition 6.2 and the structure of the spaces considered, suggests that the pathology of some examples of this section does *not* depend on the chosen triangulation. To formalize this statement we introduce the following definition.

**Definition 6.4.** Let  $\Delta$  be a simplicial complex. Denote by  $\mathcal{S}_\Delta$  and  $\mathcal{B}_\Delta$  the collection of standard decompositions of  $\Delta$  and the collection of these decompositions that are Betti splitting, respectively. We can define the *Betti splitting probability* of  $\Delta$  over  $\mathbb{F}$  as follows:

$$P_{\text{Betti}}(\Delta; \mathbb{F}) := \frac{|\mathcal{B}_\Delta|}{|\mathcal{S}_\Delta|}.$$

For a topological space  $X$  admitting a triangulation, we can define the *best Betti splitting probability* over  $\mathbb{F}$ :

$$P_{\text{Betti}}(X; \mathbb{F}) := \sup\{P_{\text{Betti}}(\Delta) : \Delta \text{ is a triangulation of } X\}.$$

Analogously we can define the *homological splitting probability*  $P_{\text{Hom}}(\Delta; \mathbb{F})$  of  $\Delta$  over  $\mathbb{F}$ . Clearly  $P_{\text{Betti}}(\Delta) \leq P_{\text{Hom}}(\Delta)$ .

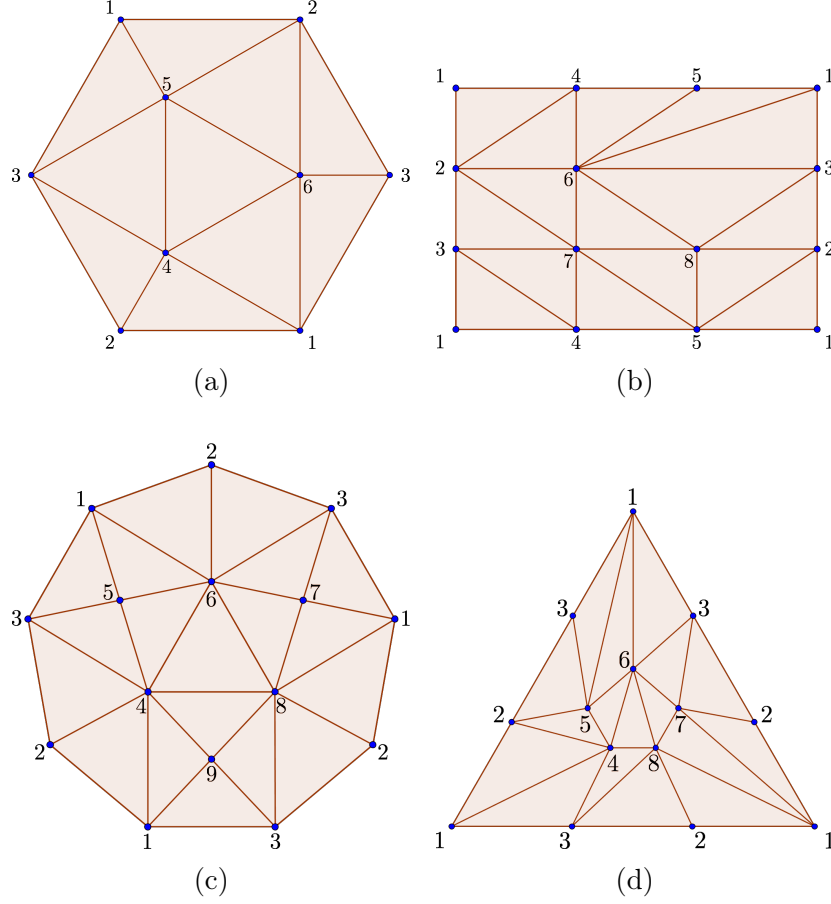


FIGURE 1. Simplicial complexes triangulating the real projective plane  $\mathbb{RP}^2$  (a), the Klein bottle  $K$  (b), the mod 3 Moore space  $M$  (c) and the Duncen hat  $D$  (d), respectively.

The next algorithm computes easily the Betti splitting probability for a simplicial complex  $\Delta$  over a field  $\mathbb{F}$ .

---

**Algorithm 4** *BettiSplittingProbability*( $\Delta, \mathbb{F}$ )

---

**Input:**  $\mathbb{F}$ , a field  
**Input:**  $\Delta$ , a simplicial complex of dimension  $d$   
 Compute all the possible standard decomposition  $\Delta_1 \cup \Delta_2$  of  $\Delta$   
 $N=0$   
**for each** standard decomposition  $\Delta_1 \cup \Delta_2$  of  $\Delta$  **do**  
   **if** *isBettiSplitting*( $\Delta, \Delta_1, \Delta_2, \mathbb{F}$ ) **then**  
      $N=N+1$   
   **end if**  
**end for**  
**return**  $P = \frac{N}{2^{|\mathcal{F}(\Delta)|}-1}$

---

In this paper, we proved that  $P_{Hom}(\Delta) = P_{Betti}(\Delta)$ , if  $\Delta$  is the triangulation of a manifold and that  $P_{Betti}(\Delta) > 0$  if  $\Delta$  is orientable. Moreover we showed, for instance,



that for the given triangulation of the Klein bottle,  $P_{Betti}(K; \mathbb{F}) = P_{Hom}(K; \mathbb{F}) = 0$ , if  $\text{char}(\mathbb{F}) \neq 2$ .

Focusing the attention only on the standard decompositions induced by the removal of a single facet, we define the *facet splitting probability* of  $\Delta$ :

$$P_{Facet}(\Delta) := \frac{|\mathcal{B}_{\mathcal{F}(\Delta)}|}{|\mathcal{F}(\Delta)|},$$

where  $\mathcal{B}_{\mathcal{F}(\Delta)}$  is the collection of standard decompositions of  $\Delta$  induced by the removal of a single facet that are Betti splitting.

**Remark 6.5.** Let  $\Delta$  be a  $d$ -manifold. By Remark 5.11, we proved that  $\Delta$  is orientable if and only if  $P_{Facet}(\Delta; \mathbb{F}) \neq 0$  for every field  $\mathbb{F}$ . In this case,  $P_{Facet}(\Delta; \mathbb{F}) = 1$ . Moreover we have that if  $\Delta$  is not orientable, then  $P_{Facet}(\Delta; \mathbb{F}) \neq 0$  if and only if  $\text{char}(\mathbb{F}) = 2$ . Also in this case  $P_{Facet}(\Delta; \mathbb{F}) = 1$ .

We are now able to state precise problems:

**Problem 1:** Let  $X$  be a non-orientable  $d$ -manifold without boundary. Is it true that  $P_{Betti}(X, \mathbb{F}) = 0$ , if  $\mathbb{F}$  is a field with  $\text{char}(\mathbb{F}) \neq 2$ ? Is it true that every triangulation of  $X$  is non-trivially decomposable?

**Problem 2:** Let  $X$  be the mod 3 Moore space. Is it true that  $P_{Betti}(X, \mathbb{F}) = 0$ , if  $\mathbb{F}$  is a field with  $\text{char}(\mathbb{F}) \neq 3$ ?

**Problem 3:** Let  $X$  be the dunce hat. Is it true that  $P_{Betti}(X, \mathbb{F}) = 0$ , for every field  $\mathbb{F}$ ?

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